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# PROBLEM OF CAPILLARY DISPLACEMENT FOR ONE MODEL OF THREE-PHASE FILTRATION

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*The class of capillary pressures corresponding to a triangular tensor of capillary diffusion in a three-phase fluid is studied. Filtration with such a tensor is described by a parabolic system of equations degenerating on solutions. This system is integrodifferential because the desired quantities are the total flow rate and the phase-saturation distribution under conditions of a specified pressure drop on the boundaries of the flow region in one of the phases. It is shown that in the problem of capillary displacement, the degenerate system can be studied using a special maximum principle.*

**Key words:** *filtration, three-phase fluid, capillary pressure, degenerate parabolic system, existence of solutions.*

**Introduction.** As shown in [1], the system of equations of three-phase filtration of immiscible incompressible fluids ignoring capillary pressures is "internally contradictory." This is due to the fact that this system is hyperbolic for some saturation values and elliptic for others. This motivates interest in models of parabolic type that take into account capillary pressure [2].

The present paper considers three-phase fluids with a triangular capillary-diffusion tensor. One-dimensional flows of such fluids are described by the system

$$u_{it} + v(t) f_i(u)_x = (B(u)_{ij} u_{jx})_x, \qquad i, j = 1, 2, \quad B_{21} = 0, \tag{1}$$

where the function of time $v(t)$ is unknown since it depends on the solution $u$. The paper gives a description of the class of capillary pressures that lead to a triangular matrix $B$.

The main feature of system (1) is that it degenerates on solutions, i.e., loses parabolicity. However, even in the case of a nondegenerate matrix $B$ there is no complete theory for systems of the form of (1). Parabolic systems with the conditions $B_{12} = B_{21} = 0$ and $B_{11} = B_{22}$ are considered in [3]. In [4], the condition $\partial f_2/\partial u_1 = 0$ is imposed in addition to the condition of boundedness of solutions and the constraints $B_{21} = 0$ and $\partial B_{22}/\partial u_1 = 0$. We note that the indicated papers ignore the constraint on the solution that follows from the physical meaning of the problem:

$$0 \leqslant u_i \leqslant 1, \qquad u_1 + u_2 \leqslant 1. \tag{2}$$

In the present paper, a theory of nondegenerate parabolic systems with constraint (2) is developed and used to solve one degenerate problem of capillary displacement.

**1. Triangular Capillary-Diffusion Tensor.** At present, there is no conventional thermodynamic principle that defines the equilibrium pressure distribution for three immiscible capillary fluids in a porous medium. Therefore, it is reasonable to consider some critical cases where pressures can be determined.

Whatever the pressure distribution in the phases, it influences only the diffusion of saturations. The constraints on the capillary-diffusion tensor stated below correspond to the critical case where the diffusion of one of the phases is determined by the saturation of this phase and does not depend on the saturations of the other two phases.

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We consider one-dimensional horizontal flows of a three-phase fluid in a porous medium in a bounded region $\Omega = \{-1 < x < 1\}$. Let $u_1$, $u_2$, and $u_3$ be the phase saturations. If the phase densities are constant, the mass balance is described by the following equations [5]:

$$\frac{\partial}{\partial t}(mu_i) + \frac{\partial}{\partial x} v_i = 0. \tag{3}$$

Here $m$ is the porosity and $v_i$ is the filtration rate of the $i$th phase. The equality

$$u_1 + u_2 + u_3 = 1 \tag{4}$$

follows from the definition of saturations as the volumetric fractions of phases. Applications use the Darcy law [5]

$$v_i = -k\lambda_i p_{ix}, \qquad \lambda_i = \lambda_i(u_1, u_2), \tag{5}$$

where $k$ is the absolute permeability and $\lambda_i$ is the mobility of the $i$th phase. For the phase pressures $p_i$, the following relations hold:

$$P_1(u_1, u_2) = p_1 - p_3, \qquad P_2(u_1, u_2) = p_2 - p_3.$$

Here the capillary pressures $P_i$ are considered specified functions of $u_1$ and $u_2$.

We denote

$$\lambda = \sum_1^3 \lambda_i, \qquad f_i = \frac{\lambda_i}{\lambda}, \qquad v = \sum_1^3 v_i, \qquad i = 1, 2, 3. \tag{6}$$

Then, from Eqs. (3) and (5) it follows that $v_x = 0$, i.e., $v$ depends only on $t$.

Summation of Eqs. (5) yields the equality

$$-\frac{\partial p_3}{\partial x} = \frac{v}{\lambda} + \frac{\lambda_1}{\lambda}\frac{\partial P_1}{\partial x} + \frac{\lambda_2}{\lambda}\frac{\partial P_2}{\partial x}. \tag{7}$$

If introduce the notation $p_3(1,t) - p_3(-1,t) = \Delta p_3$, integration of equality (7) leads to the following representation for the total velocity:

$$v(t) = -\Big(\Delta p_3 + \int_{-1}^{1}\Big(\frac{\lambda_1}{\lambda}\frac{\partial P_1}{\partial x} + \frac{\lambda_2}{\lambda}\frac{\partial P_2}{\partial x}\Big)\,dx\Big)\Big/\int_{-1}^{1}\lambda^{-1}\,dx. \tag{8}$$

Therefore, equalities (5) are written as

$$v_1 = vf_1 + \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_2}{\partial x} - \frac{\lambda_1(\lambda_2 + \lambda_3)}{\lambda}\frac{\partial P_1}{\partial x},$$

$$v_2 = vf_2 + \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_1}{\partial x} - \frac{\lambda_2(\lambda_1 + \lambda_3)}{\lambda}\frac{\partial P_2}{\partial x}.$$

Thus, from Eq. (3) we obtain a closed system of equations for the functions $u_1$ and $u_2$:

$$u_{it} + v(t) f_i(u)_x = (B(u)_{ij} u_{jx})_x. \tag{9}$$

Here $f_j(u_1, u_2)$ and $v(t)$ are defined by formulas (6) and (8) and the matrix $B$ has the form

$$\begin{aligned} B_{11} &= \frac{\lambda_1(\lambda_2+\lambda_3)}{\lambda}\frac{\partial P_1}{\partial u_1} - \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_2}{\partial u_1}, \qquad & B_{12} &= -\frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_2}{\partial u_2} + \frac{\lambda_1(\lambda_2+\lambda_3)}{\lambda}\frac{\partial P_1}{\partial u_2}, \\ B_{21} &= \frac{\lambda_2(\lambda_1+\lambda_3)}{\lambda}\frac{\partial P_2}{\partial u_1} - \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_1}{\partial u_1}, \qquad & B_{22} &= -\frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_1}{\partial u_2} + \frac{\lambda_2(\lambda_1+\lambda_3)}{\lambda}\frac{\partial P_2}{\partial u_2}. \end{aligned} \tag{10}$$

The nonlocal system (9) describes motion under the action of a pressure drop in one of the phases on the boundaries of the flow region.

From condition (4) follows the constraint

$$u \in \Delta = \{u\colon \quad u \in \mathbb{R}^2, \quad 0 \leqslant u_i \leqslant 1, \quad u_1 + u_2 \leqslant 1\}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Note that the triangle $\Delta$ can be treated as the intersection of the half-planes:

$$\Delta = \bigcap_1^3 \{G_i(u) \leqslant 0\}, \qquad G_1 = -u_1, \quad G_2 = -u_2, \quad G_3 = u_1 + u_2 - 1. \tag{11}$$

We formulate the following assumptions on the empirical parameters (the functions $\lambda_i$ and $P_i$):

$$\lambda_i = \lambda_i(u_i) \geqslant 0, \qquad \lambda_i|_{u_i=0} = 0, \qquad i \in \{1,2,3\}; \tag{12}$$

$$B_{21} = 0, \qquad B_{11} \geqslant 0, \qquad B_{22} = B_{22}(u_2) \geqslant 0 \qquad \text{in} \quad \Delta. \tag{13}$$

Condition (12) is conventional; it is justified, for example, in [5]. This condition, in particular, leads to degeneration of system (9). Condition (13) implies that the first and third phases do not determine the diffusion process in the second phase [6, 7]. The equalities $B_{21} = 0$ and $B_{22} = B_{22}(u_2)$ represent the *capillary diffusion hypothesis* and can be written as

$$A\,\frac{\partial P_1}{\partial u_1} = \frac{\partial P_2}{\partial u_1}, \qquad \frac{\partial P_2}{\partial u_2} = A\,\frac{\partial P_1}{\partial u_2} + \frac{\lambda B_{22}(u_2)}{\lambda_2(\lambda_1+\lambda_3)}, \qquad A = \frac{\lambda_1}{\lambda_1+\lambda_3}. \tag{14}$$

By virtue of conditions (14), system (9) is simplified. Indeed, the following formulas are valid:

$$\frac{\lambda_1}{\lambda}\,\frac{\partial P_1}{\partial u_1} + \frac{\lambda_2}{\lambda}\,\frac{\partial P_2}{\partial u_1} = \frac{\partial P_2}{\partial u_1}, \qquad \frac{\lambda_1}{\lambda}\,\frac{\partial P_1}{\partial u_2} + \frac{\lambda_2}{\lambda}\,\frac{\partial P_2}{\partial u_2} = -\frac{B_{22}}{\lambda_2} + \frac{\partial P_2}{\partial u_2}.$$

Therefore,

$$\frac{\lambda_1}{\lambda}\,\frac{\partial P_1}{\partial x} + \frac{\lambda_2}{\lambda}\,\frac{\partial P_2}{\partial x} = \frac{\partial P_2}{\partial x} + \frac{\partial F}{\partial x}, \qquad F(u_2) = -\int_0^{u_2} \frac{B_{22}(s)}{\lambda_2(s)}\,ds.$$

Then, formula (8) can be written as

$$v(t) = -(p_2 + F(u_2)) \Big/ \int_{-1}^{1} \lambda^{-1}\,dx \Big|_{-1}^{1}.$$

We assume that the difference $\Delta p_2 = p_2\Big|_{x=-1}^{x=1}$ and the saturations $u_2$ at the terminal points $x = \pm 1$ are specified functions of time. Therefore, in system (9), $v(t)$ is a functional of the solution of the form

$$v(t) = g_1(t) \Big/ \int_{-1}^{1} æ(u_1,u_2)\,dx \equiv g[t;u], \qquad g_1 = -\Delta p_2(t) - \Delta F(t), \qquad æ = \lambda^{-1},$$

where

$$\Delta F(t) = F(u_2(1,t)) - F(u_2(-1,t)).$$

The condition of capillary displacement is the equality $\sum\limits_1^3 v_i = 0$, which implies that the motion of one of the phases is opposite to the motions of the other two phases [8]. In particular, $v(t) \equiv 0$ if

$$p_2\Big|_{x=-1} = p_2\Big|_{x=1}, \qquad u_2\Big|_{x=-1} = u_2\Big|_{x=1}. \tag{15}$$

From the derivation of the nonlocal system (9), it follows that the capillary-displacement conditions (15) lead to the simplified system

$$u_{it} = (B(u)_{ij} u_{jx})_x, \qquad u \in \Delta. \tag{16}$$

We note that these equations are formally separated because the second equation does not contain the function $u_1$. At the same time, by virtue of the condition $u_2(t,x) \leqslant 1 - u_1(t,x)$, the second equation cannot be solved separately from the first.

**2. Mobilities and Capillary Pressures.** For specified mobilities $\lambda_i(u_1, u_2)$ $(i = 1, 2, 3)$ and the coefficient $B_{22}(u_2)$, conditions (14) are a system of linear equations for the capillary pressures $P_i(u_1, u_2)$. We analyze this system in the case of practical importance where the mobilities are homogeneous functions [5]:

$$\lambda_i = k_i u_i^n, \qquad k_i = \text{const} > 0, \qquad n > 0. \tag{17}$$

We first consider the homogeneous system with $B_{22} = 0$ and apply an algorithm for calculating symmetry groups [9]. If the homogeneous system (14) admits a one-parameter group with the infinitesimal operator

$$X = \zeta^1(u_1, u_2, P_1, P_2)\,\frac{\partial}{\partial u_1} + \zeta^2(\cdots)\,\frac{\partial}{\partial u_2} + \eta^1(\cdots)\,\frac{\partial}{\partial P_1} + \eta^2(\cdots)\,\frac{\partial}{\partial P_2},$$

then the functions $\zeta^i$ and $\eta^i$ satisfy the constraints

$$\zeta^1\,\frac{\partial A}{\partial u_1} + \zeta^2\,\frac{\partial A}{\partial u_2} + A\Big(\frac{\partial \eta^1}{\partial P_1} + A\,\frac{\partial \eta^1}{\partial P_2}\Big) = \frac{\partial \eta^2}{\partial P_1} + A\frac{\partial \eta^2}{\partial P_2},$$

$$\frac{\partial \eta^2}{\partial u_2} = A\,\frac{\partial \eta^1}{\partial u_2}, \qquad \frac{\partial \eta^2}{\partial u_1} = A\,\frac{\partial \eta^1}{\partial u_1}.$$

From the given conditions it follows, in particular, that the system admits a group with the nontrivial operator

$$X = -\xi\,\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}, \qquad \xi = \frac{u_1}{1 - u_2}.$$

This implies that for any number $a$, the transformation of variables $(u_1, u_2) \to (u_1', u_2')$, where

$$u_1' = u_1 - a u_1/(1 - u_2), \qquad u_2' = u_2 + a \qquad (a \in \mathbb{R}),$$

converts any solution to a certain solution of the same homogeneous system. By virtue of conditions (17), the function $A(u)$ depends only on the variable $\xi$. It is easy to show that a pair of functions $P_1 = \varphi(\xi)$ and $P_2 = \phi(\xi)$ is a solution of the homogeneous system (14) if $\phi'(\xi) = A(\xi)\varphi'(\xi)$. We seek a solution of the inhomogeneous system (14) in the case where

$$B_{22} = \alpha u_2^n (1 - u_2)^n, \qquad \alpha = \text{const} \geqslant 0.$$

The solution is sought in the form

$$P_1 = a(u_2)b(\xi) + \varphi(\xi), \qquad P_2 = a(u_2)B(\xi) + c(u_2) + \phi(\xi), \qquad B' = Ab'. \tag{18}$$

Substitution of these formulas into (14) yields

$$a'(u_2) = \frac{\alpha u_2^n}{k_1}, \qquad b(\xi) = \frac{k_1}{k_3(1 - \xi)^{n-1}} - \frac{1}{\xi^{n-1}},$$
$$B(\xi) = \frac{k_1}{k_3(1 - \xi)^{n-1}}, \qquad c'(u_2) = \frac{\alpha(1 - u_2)^n}{k_2}. \tag{19}$$

Next, from formulas (10), we determine the capillary diffusion tensor $B$:

$$B_{11} = k_3(\varphi'(\xi) + a(u_2)b'(\xi))A(\xi)(1 - \xi)^n (1 - u_2)^{n-1},$$
$$B_{22} = \alpha u_2^n (1 - u_2)^n, \qquad B_{12} = \xi(B_{11} - B_{22}), \qquad B_{21} = 0. \tag{20}$$

Thus, if the mobilities and capillary pressures are specified by formulas (17)–(19), the matrix $B$ is triangular and has the form of (20). Inequalities (13) are satisfied if $\varphi'(\xi) \geqslant 0$.

**3. Maximum Principle.** For system (9), the maximum principle, known as *the principle of positively invariant regions*, holds [10]. Let the saturations $u_i$ $(i = 1, 2)$ satisfy the initial and boundary conditions

$$u_i(0, x) = u_{i0}(x), \qquad u_i(t, \pm 1) = u_{i\pm}(t), \qquad u_0(x) \in \Delta, \quad u_\pm(t) \in \Delta. \tag{21}$$

The indicated maximum principle is formulated as follows: $u(t, x) \in \Delta$ for any $(t, x) \in Q = (0, T) \times \Omega$. Below, we shall prove the inclusion of $u \in \Delta$, but we first verify satisfaction of the following two conditions necessary for this [10]:

$$B^{\mathrm{t}}\langle \nabla_u G_i \rangle = \mu_i \nabla_u G_i, \qquad (f')^{\mathrm{t}}\langle \nabla_u G_i \rangle = \alpha_i \nabla_u G_i \qquad \text{if} \quad G_i(u) = 0. \tag{22}$$

Here the matrix $B$ and the vector $f$ are specified by formulas (20), (6), and (17). Conditions (22) imply that the normal vector to the boundary $G_i(u) = 0$ of the triangle $\Delta$ is an eigenvector of the matrices $B^{\mathrm{t}}$ and $f'^{\mathrm{t}}$ on the boundary $G_i(u) = 0$, where elements of the matrix $f'$ are the numbers $\partial f_i/\partial u_j$. In this case, the eigenvalues $\mu_i$ should be nonnegative.

If the functions $G_i(u)$ are defined by formulas (11), the first equality in (22) is equivalent to the three equalities

$$B_{12} = 0, \qquad B_{21} = 0, \qquad B_{11} = B_{12} + B_{22}$$

on the segments $G_1 = 0$, $G_2 = 0$, and $G_3 = 0$, respectively. These equalities are valid by virtue of the definitions of the coefficients $B_{ij}$.

We check equality (22) for $f'$ on the segment $G_3(u) = 0$. The vector $\nabla_u G_3$ is an eigenvector of the matrix $(f')^{\mathrm{t}}$ only if

$$\frac{\partial f_1}{\partial u_1} + \frac{\partial f_2}{\partial u_1} = \frac{\partial f_1}{\partial u_2} + \frac{\partial f_2}{\partial u_2} \qquad \text{at} \quad u_1 + u_2 = 1.$$

At the same time, this equality is a simple corollary of condition (12) for $\lambda_3$:

$$\lambda_3(u_1, u_2) = 0 \qquad \text{at} \quad u_1 + u_2 = 1.$$

The other two conditions on the matrix $f'$ are similarly checked.

For system (16), describing capillary displacement, another maximum principle also holds. This principle permits one to prove the resolvability of the Dirichlet problem (21) subject to the conditions

$$0 < \delta \leqslant u_{i0}(x) \leqslant 1 - \delta, \qquad 0 < \delta \leqslant u_{i\pm}(t) \leqslant 1 - \delta. \tag{23}$$

We show that there exists a number $0 < \delta' \leqslant \delta$ such that

$$0 < \delta' \leqslant u_i(t, x) \leqslant 1 - \delta'. \tag{24}$$

Indeed, in terms of the functions $\xi$ and $u_2$, system (16) is written as

$$\xi_t = (B_{11}\xi_x)_x - \frac{\xi_x u_{2x}(B_{11} + B_{22})}{1 - u_2}, \qquad u_{2t} = (B_{22}(u_2)u_{2x})_x.$$

For each of these equations, the conventional maximum principle holds [3]. Under conditions (23), there exists $\tilde{\delta} > 0$ such that $\tilde{\delta} \leqslant \xi \leqslant 1 - \tilde{\delta}$ at $t = 0$ and $x = \pm 1$. It is now obvious that $\tilde{\delta} \leqslant \xi \leqslant 1 - \tilde{\delta}$ for all $(t, x) \in Q$. This estimate combined with the apparent estimate $\delta \leqslant u_2 \leqslant 1 - \delta$ are equivalent to estimates (24).

**4. Approximate Solutions.** For the problem (9), (21), we construct approximate solutions that depend on the parameters $\varepsilon$, $\nu$, and $\delta$. Let $h$: $\mathbb{R}^2 \to \mathbb{R}^2$ be a smooth function that satisfies the inequality $\nabla_u G_i \cdot h(u) < 0$ $(i = 1, 2, 3)$ near the boundary $\partial\Delta$ of the triangle $\Delta$.

In the present section, the following smoothness of the input data is assumed:

$$u_0(x) \in H^{2+\beta}(\overline{\Omega}), \qquad u_\pm(t) \in H^{1+\beta}([0, T]), \qquad 0 < \beta < 1.$$

We consider the problem

$$u_t + g[t; u]f(u)_x = (D^\nu u_x)_x + \varepsilon h, \qquad (t, x) \in Q; \tag{25}$$

$$\delta u_n + u = u_{\partial\varepsilon} \quad \text{at} \quad |x| = 1, \qquad u\Big|_{t=0} = u_{0\varepsilon}(x). \tag{26}$$

Here

$$u_n\Big|_{x=\pm 1} = \pm u_x, \qquad u_\partial\Big|_{x=\pm 1} = u_\pm(t),$$

$$u_{i\partial\varepsilon} = (1 - \varepsilon)(\varepsilon/2 + u_{i\partial}), \qquad u_{i0\varepsilon} = (1 - \varepsilon)(\varepsilon/2 + u_{i0}), \qquad i = 1, 2.$$

The smooth matrix $D^\nu$ satisfies the conditions

$$D^\nu \geqslant \nu, \quad D^\nu_{21} = 0, \qquad \Big(D^{\nu\mathrm{t}}\langle\nabla_u G_i\rangle - \mu_i \nabla_u G_i\Big)\Big|_{G_i(u)=0} = 0, \qquad \mu_i \geqslant 0. \tag{27}$$

Generally, the vector function $f(u)$ is defined by formulas (6) and (12). Below in this section, the superscripts $\nu$ and superscript $\varepsilon$ are omitted for simplicity.

In the case of periodic boundary conditions and for $g[t;u] \equiv 1$, the nondegenerate problem (25), (26) was examined in [6]. In the present paper, because we use the same method, we examine the problem (25), (26) schematically, outlining the main steps and singularities.

**Step 1.** We formulate the following statement.

**Lemma 4.1.** *The values of a solution $u(t,x)$ lie strictly inside the triangle $\Delta$ if $u_0(x), u_\pm(t) \in \Delta$ for all $x$ and $t$.*

**Proof.** Following the method of positively invariant regions, we designate $z^i = G_i(u)$. We prove that $z^i < 0$ for each $i$. By the conditions of the lemma,

$$\max_{x\in\Omega} z^i(0,x) < 0, \qquad i \in \{1,2,3\}.$$

We assume that there exists the first time $t_1 > 0$ such that

$$\max_{x\in\Omega} z^i(t_1,x) = z^i(t_1,x_0) = 0$$

for a certain $i$. There is an alternative: either $|x_0| < 1$ or $|x_0| = 1$. The case $x_0 = 1$ is impossible. Indeed, since $\delta z^i_x + z^i = -u^i_+$, it follows that $z^i_x(t_1,1) < 0$. By virtue of continuity, there exists a time $t_0 \in (0,t_1)$ such that $\max z^i(t_0,x) = 0$ and the maximum is taken over all $x \in \Omega$. This contradicts the choice of $t_1$. The impossibility of satisfaction of the equality $x_0 = -1$ is established similarly.

Let us consider the case $|x_0| < 1$. We multiply Eq. (25) by $\nabla_u G_i$; then, in view of conditions (27), the following equality holds at the point $(t_1,x_0)$:

$$z^i_t + g[t;u]\alpha_i z^i_x = (\mu_i z^i_x)_x + \varepsilon h\cdot\nabla_u G_i. \tag{28}$$

It is assumed that $z^i(t_1,x_0) = \max z^i(\tau,y)$, and the maximum is taken over all $0 \leqslant \tau \leqslant t_1$ and $|y| \leqslant 1$. Therefore,

$$z^i_x(t_1,x_0) = 0, \qquad z^i_{xx}(t_1,x_0) \leqslant 0, \qquad z^i_t(t_1,x_0) \geqslant 0. \tag{29}$$

The inequality $h\cdot\nabla_u G_i < 0$ is satisfied at the point $(t_1,x_0)$ by virtue of the choice of the function $h$. Then, from Eq. (28) follows the inequality $z^i_t(t_1,x_0) < 0$, which contradicts (29).

**Step 2.** The following estimate holds:

$$\|u_x\|_{L^2(Q)} + \delta\sum_{\pm}\int_0^T |u_x(t,\pm1)|^2\,dt \leqslant c. \tag{30}$$

Here the constant $c$ depends on $\|\dot u_\pm\|_{L^1(0,T)}$, $\|h\|_{L^1(Q)}$, and $\nu$ and does not depend on $\varepsilon$ and $\delta$. For the function $u_2$, this estimate follows from the equality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_2^2\,dx + \int_\Omega D_{22}|v_{2x}|^2\,dx = v_2(D_{22}(v_{2x}+w_{2x}) - gf_2)\Big|_{-1}^{+1} + \int_\Omega \Big(v_{2x}(gf_2 - D_{22}w_{2x}) - w_{2t}v_2 + \varepsilon h_2 v_2\Big)\,dx,$$

in which

$$w = (1-x)u_-/2 + (1+x)u_+/2, \qquad v = u - w.$$

Next, a similar equality is obtained for $v_1$ by multiplying the first equation of system (25) by $u_1$, and thereby, the estimate (30) for $u_1$ is derived.

As a corollary of Eqs. (25), we obtain the following estimate which is uniform in $\varepsilon$ and $\delta$:

$$\|u_t\|_{L^2(0,T;W^{-1,2}(\Omega))} \leqslant c. \tag{31}$$

**Step 3.** For a certain constant $\alpha \in (0,1)$, the following estimate holds:

$$|u_2|_Q^{(\alpha)} \equiv \|u_2\|_{H^{\alpha,\alpha/2}(\overline{Q})} \leqslant c. \tag{32}$$

Here and below, the constant $c$ depends on $\delta$.

The proof is based on the reduction method known in the theory of linear parabolic equations [3]. We introduce a smooth function $0 \leqslant \zeta(t,x) \leqslant 1$ that is different from zero only for $x \in K_\rho$ ($K_\rho$ is an open sphere of radius $\rho$ with center at $x^0 \in \overline{\Omega}$). We designate

$$\Omega_\rho = \overline{\Omega}\cap K_\rho = [x^0_-, x^0_+], \qquad x^0_+ = \min\{1, x^0+\rho\}, \qquad x^0_- = \max\{-1, x^0-\rho\}.$$

Multiplication of the second equation of system (25) by the function

$$\zeta^2 \max\{u_2 - k, 0\} \equiv \zeta^2 u_2^{(k)} \qquad (k \in \mathbb{R})$$

and subsequent integration over $\Omega_\rho$ yield the equality

$$\frac{1}{2}\,\frac{d}{dt}\int\limits_{\Omega_\rho} \zeta^2 |u_2^{(k)}|^2\,dx + \int\limits_{\Omega_\rho} \zeta^2 D_{22} |u_{2x}^{(k)}|^2\,dx = \zeta^2 D_{22} u_{2x} u_2^{(k)}\Big|_{x_-^0}^{x_+^0} - \zeta^2 g f_2 u_2^{(k)}\Big|_{x_-^0}^{x_+^0}$$

$$- \int\limits_{\Omega_\rho} \Big(2\zeta\zeta_x D_{22} u_{2x} u_2^{(k)} - \zeta\zeta_t |u_2^{(k)}|^2 - g f_2 (2\zeta\zeta_x u_2^{(k)} + \zeta^2 u_{2x}^{(k)}) - \varepsilon h_2 \zeta^2 u_2^{(k)}\Big)\,dx.$$

We note that $D_{22} \geqslant \nu$ and $\delta u_x\Big|_{x=\pm 1} = \pm(u_{\partial\varepsilon\pm} - u)\Big|_{x=\pm 1}$; for small $\rho$, we have

$$\zeta^2 D_{22} u_{2x} u_2^{(k)}\Big|_{x_-^0}^{x_+^0} \leqslant \frac{1}{\delta}\,\zeta^2 D_{22} u_2^{(k)} u_{2+}\Big|_{x=1} + \frac{1}{\delta}\,\zeta^2 D_{22} u_2^{(k)} u_{2-}\Big|_{x=-1},$$

$$|\zeta^2 v^{(k)}|_{|x|=1} \leqslant \Big|\int\limits_{\Omega_\rho} (\zeta^2 v_x^{(k)} + 2\zeta\zeta_x v^{(k)})\,dx\Big|.$$

Therefore,

$$\frac{1}{2}\,\frac{d}{dt}\int\limits_{\Omega_\rho} \zeta^2 |u_2^{(k)}|^2\,dx + \nu\int\limits_{\Omega_\rho} \zeta^2 |u_{2x}^{(k)}|^2\,dx \leqslant \frac{\nu}{2}\int\limits_{\Omega_\rho} \zeta^2 |u_{2x}^{(k)}|^2\,dx + c_1\int\limits_{\Omega_\rho} \Big(|u_2^{(k)}|^2 (|\zeta_x|^2 + |\zeta\zeta_t| + \zeta^2) + \zeta^2 \mathbf{1}_{A_{k,\rho}(t)}\Big)\,dx,$$

where $A_{k,\rho}(t)$ is the intersection of the support $u_2^{(k)}$ with the sphere $K_\rho$; the designation $\mathbf{1}_A$ is used for the characteristic function of the set $A$. The last inequality implies that $u_2$ belongs to the class $\mathcal{B}_2(Q, M, \gamma, r, \delta, k)$, where $r = 6$ (see formula (7.5) in [3, chapter 2, § 7]). Therefore, $u_2 \in H^{\alpha,\alpha/2}(\overline{Q})$ for a certain $\alpha \in (0, 1)$.

**Step 4.** The following estimate holds:

$$\max_{0\leqslant t\leqslant T}\Big\{\int\limits_{\Omega} u_{2x}^2\,dx + \delta\sum_{x=\pm 1} u_{2x}^2\Big\} + \int\limits_{Q} (u_{2xx}^2 + u_{2x}^4 + u_{2t}^2)\,dx\,dt \leqslant c.$$

Its derivation is based on multiplication of the second equation of system (25) by the function $\zeta(x)$ which possesses the above-mentioned properties. In this case, we use the well-known inequality [3]

$$\int\limits_{K_\rho} \zeta^2 v_x^4 dx \leqslant 16\mathrm{osc}^2\{v, K_\rho\} \int\limits_{K_\rho} (2\zeta^2 v_{xx}^2 + \zeta^2 v_x^2)\,dx$$

and estimate

$$\mathrm{osc}^2\{u_2, K_\rho\} \leqslant c\rho^{\alpha_1} \qquad (\alpha_1 < \alpha),$$

which follows from (32).

**Step 5.** The following estimates hold: $|u_1|_Q^{(\alpha)} \leqslant c$ and

$$\max_{0\leqslant t\leqslant T}\Big\{\int\limits_{\Omega} u_{1x}^2\,dx + \delta\sum_{x=\pm 1} |u_{1x}|^2\Big\} + \int\limits_{Q} (u_{1xx}^2 + u_{1x}^4 + u_{1t}^2)\,dx\,dt \leqslant c.$$

This is proved similarly to the case of the function $u_2$, taking into account the estimates obtained before.

**Step 6.** The following estimates hold:

$$\int\limits_{Q} |u_{ix}|^6\,dx\,dt \leqslant c, \qquad \int\limits_{Q} |u_{ix} u_{ixx}|^2\,dx\,dt \leqslant c.$$

By virtue of the inequalities

$$\int\limits_{Q} |u_{ix}|^6\,dx\,dt \leqslant \int\limits_{0}^{T} \max_{x\in\Omega} |u_{ix}|^4 \int\limits_{\Omega} |u_{ix}|^2\,dx\,dt, \qquad \max_{|x|\leqslant 1} v_x^4 \leqslant (1/2)\|v_x\|_{L^2(\Omega)}^4 + 8\|v_{xx}\|_{L^2(\Omega)}^2 \|v_x\|_{L^2(\Omega)}^2,$$

we have

$$\|u_{ix}\|_{L^6(Q)}^6 \leqslant (1/2)\|u_{ix}\|_{L^\infty(0,T;L^2(\Omega))}^4 (1 + \|u_{ixx}\|_{L^2(Q)}^2) \leqslant c.$$

Thus, the first of the formulated estimates is proved. The second estimate for the function $u_2$ is proved as follows. The second equation of system (25) can be treated as a linear equation for the function $u_2$ and written as

$$u_{2t} = D_{22}u_{2xx} + F,$$

$$F = \frac{\partial D_{22}}{\partial u_1}\, u_{1x}u_{2x} + \frac{\partial D_{22}}{\partial u_2}\, |u_{2x}|^2 - g[t;u]\Big(\frac{\partial f_2}{\partial u_1}\, u_{1x} + \frac{\partial f_2}{\partial u_2}\, u_{2x}\Big) + \varepsilon h_2.$$

From the aforesaid, it follows that $\|F\|_{L^3(Q)} \leqslant c$. Therefore, according to the theory of linear equations [3], we have

$$\int\limits_Q |u_{2xx}|^3\, dx\, dt \leqslant c.$$

Then, the second estimate for the function $u_2$ follows from the inequality

$$\int\limits_Q |uv|^2\, dx\, dt \leqslant \Big(\int\limits_Q |u|^6\, dx\, dt\Big)^{1/3}\Big(\int\limits_Q |v|^3\, dx\, dt\Big)^{2/3}.$$

The function $u_1$ is similarly considered.

**Step 7.** There exists a constant $\alpha \in (0,1)$ such that $|u_{ix}|_Q^{(\alpha)} \leqslant c$.

To prove these estimates, we use boundary condition (26), which allows us to consider the derivatives $u_{ix}$ bounded on the boundary of the region $\Omega$ and to act as in the case of the theory of linear parabolic equations. Indeed, the function $v = u_{2x}$ is a solution of the equation

$$v_t = (D_{22}(u_2)v_x)_x + F + H_x,$$

$$F = \frac{\partial D_{22}}{\partial u_1}\, u_{1x}u_{2xx} + \frac{\partial D_{22}}{\partial u_1}\, u_{1xx}u_{2x} + 2\,\frac{\partial D_{22}}{\partial u_2}\, u_{2x}u_{2xx} + \frac{\partial^2 D_{22}}{\partial u_1^2}\, u_{1x}^2u_{2x} + 2\,\frac{\partial^2 D_{22}}{\partial u_1\partial u_2}\, u_{1x}u_{2x}^2 + \frac{\partial^2 D_{22}}{\partial u_2^2}\,(u_{2x})^3,$$

$$H = -g[t;u]\,\frac{\partial f_2}{\partial u_1}\, u_{1x} - g[t;u]\,\frac{\partial f_2}{\partial u_2} u_{2x} + \varepsilon h_2.$$

On the strength of the estimates obtained above, we have

$$\|F\|_{q,r,Q} \equiv \Big(\int\Big(\int\limits_\Omega F^q\Big)^{r/q} dt\Big)^{1/r} \leqslant c, \qquad \|H^2\|_{q,r,Q} \leqslant c$$

for $q = 2$ and $r = 2$. The constants $q$ and $r$ satisfy the conditions

$$\frac{1}{r} + \frac{1}{2q} = 1 - æ, \quad 0 < æ < \frac{1}{2}, \quad q \in [1,\infty], \quad r \in \Big[\frac{1}{1-æ}, \frac{2}{1-2æ}\Big], \quad æ = \frac{1}{4}.$$

From the boundary conditions, we have $\|v(t,\pm1)\|_{H^{\alpha/2}([0,T])} \leqslant c$, $\alpha \leqslant \beta$. According to the linear theory [3, chapter 3, § 10], the estimate $|v|_Q^{(\alpha)} \leqslant c$ holds for a certain $\alpha$. The estimate for the function $u_1$ is similarly proved.

Next, it is assumed that the initial and boundary data in (26) satisfy the matching conditions

$$\pm\delta u_0'(\pm1) + u_0(\pm1) = u_\pm(0). \tag{33}$$

Under these conditions, we derive the a prior estimate

$$|u|_Q^{(2+\beta)} \leqslant c, \tag{34}$$

which is uniform in $\varepsilon$. Then, just as in [6], the Leray–Schauder theorem on a fixed point is applied and the existence and uniqueness of approximate solutions are proved.

**Theorem 4.1.** *Let the functions $f(u)$, $\nabla_u f$, $D_{ij}(u)$, $\nabla_u D_{ij}$, $\partial^2 D_{ij}/\partial u_i\partial u_j$, $g_1(t)$, and $h(u)$ satisfy the condition of Hölder continuity with the exponent $\beta \in (0,1)$ and the matching conditions* (33) *hold. Then, the problem* (25), (26) *has a single solution $u(t,x) \in H^{2+\beta,1+\beta/2}(\bar{Q})$ such that $\boldsymbol{u}(t,x) \in \Delta$ for all $(t,x) \in Q$.*

**Corollary 1.** Since the estimate (34) does not depend on $\varepsilon$, Theorem 4.1 also holds true for $\varepsilon = 0$ as well.

**5. Degenerate Problem of Capillary Displacement.** In the present section, we examine system (16) subject to the conditions

$$u\Big|_{x=\pm 1} = u_\pm(t), \qquad u\Big|_{t=0} = u_0(x). \tag{35}$$

Let us consider the approximate nondegenerate problem

$$u_t = (D^\nu(u)u_x)_x,$$

$$\nu u_n + u = u_\partial^\nu \quad \text{at} \quad |x| = 1, \qquad u|_{t=0} = u_0^\nu(x).$$

Here

$$D_{11}^\nu = \nu + \chi_\nu(u_2)B_{11}, \quad B_{22}^\nu = \nu + \chi_\nu(u_2)B_{22}, \quad B_{12}^\nu = \chi_\nu(u_2)\xi(B_{11} - B_{22}),$$

$$B_{21}^\nu = 0, \quad u_0^\nu \in H^{2+\beta}(\overline{\Omega}), \quad u_0^\nu(x) \in \Delta, \quad u_\pm^\nu \in H^{1+\beta/2}([0,T]), \quad u_\pm^\nu(t) \in \Delta,$$

$$\pm\nu u_0^\nu(\pm 1) + u_0^\nu(\pm 1) = u_\pm^\nu(0),$$

$$\|u_\pm^\nu - u_\pm\|_{W^{1,1}(0,T)} \to 0, \quad \|u_0^\nu - u_0\|_{L^2(\Omega)} \to 0 \quad \text{for} \quad \nu \downarrow 0,$$

where $\chi_\nu(u_2)$ is a smooth function:

$$\chi_\nu(u_2) = 1 \quad \text{if} \quad 0 \leqslant u_2 \leqslant 1 - \nu \quad \text{or} \quad \chi_\nu(u_2) = 0 \quad \text{if} \quad 1 - \nu/2 \leqslant u_2 \leqslant 1.$$

The function $\xi(u_1, u_2)$ is discontinuous at the point $(0, 1)$; therefore, we introduce the function $\chi_\nu(u_2)$ to ensure regularity in the triangle $\Delta$ of the matrix $B^\nu$ [the matrix $B$ is specified by formulas (20)].

Let the nondegeneracy conditions (23) for the initial and boundary data be satisfied. As shown above, the problem (16), (35) has a single classical solution $u^\nu$ with the apparent estimate $\delta \leqslant u_2^\nu(t,x) \leqslant 1 - \delta$. Therefore, for sufficiently small $\nu$, the following equalities hold:

$$D_{11}^\nu = \nu + B_{11}, \qquad D_{22}^\nu = \nu + B_{22}, \qquad D_{12}^\nu = \xi(D_{11}^\nu - D_{22}^\nu). \tag{36}$$

Since $D_{22}^\nu(u^\nu) \geqslant \delta_1 > 0$ is uniform in $\nu$, relations (30) and (31) imply the following estimates, which are uniform in $\nu$:

$$\|u_{2x}^\nu\|_{L^2(Q)} \leqslant c, \qquad \|u_{2t}^\nu\|_{L^2(0,T;W^{-1,2}(\Omega))} \leqslant c. \tag{37}$$

In view of formulas (36), the function $\xi = u_1^\nu/(1 - u_2^\nu)$ is a solution of the problem

$$\xi_t = (D_{11}^\nu \xi_x)_x - \xi_x u_{2x}^\nu (D_{11}^\nu + D_{22}^\nu)/(1 - u_2^\nu),$$

$$\frac{\nu(1 - u_2^\nu)}{1 - u_\pm^\nu}\,\xi_n + \xi = \xi_\pm \qquad \text{at} \quad x = \pm 1, \quad \xi\Big|_{t=0} = \xi_0(x).$$

By virtue of conditions (23), $\delta \leqslant \xi(t,x) \leqslant 1 - \delta$ is uniform in $\nu$, according to the maximum principle. Obviously, the following estimates uniform in $\nu$ hold:

$$\delta^2 \leqslant u_1^\nu(t,x) \leqslant (1-\delta)^2, \qquad D_{11}^\nu \geqslant \delta_2 > 0.$$

From (30) and (31) it follows that

$$\|u_{1x}^\nu\|_{L^2(Q)} \leqslant c, \qquad \|u_{1t}^\nu\|_{L^2(0,T;W^{-1,2}(\Omega))} \leqslant c \tag{38}$$

are uniform in $\nu$.

According to the Aubin–Lions theorem, estimates (37) and (38) imply the existence of a certain sequence of solutions $u^n \equiv u^{\nu_n}$ and a limiting vector function $u$ such that $u^n(t,x) \to u(t,x)$ almost everywhere in $Q$ and $u_x^n \to u_x$ weakly in $L^2(Q)$. Thus, it is proved that the problem (16), (35) has a weak solution. This result can be formulated as the following theorem.

**Theorem 5.1.** *Let the matrix $B$ satisfy the conditions of theorem* 4.1 *and let the initial and boundary data satisfy conditions* (23) *and $u_{i\pm}(t) \in W^{1,1}(0,T)$. Then, the problem* (16), (35) *has a solution $(u_1, u_2)$ such that*

$$u \in L^\infty(Q) \cap L^2(0,T;W^{1,2}(\Omega)), \qquad u_t \in L^2(0,T;W^{-1,2}(\Omega))$$

*and*

$$\int\limits_Q \Big(u_i\varphi_t - B_{ij}(u)\,\frac{\partial u_j}{\partial x}\,\frac{\partial \varphi}{\partial x}\Big)\,dx\,dt + \int\limits_\Omega u_{i0}(x)\varphi(0,x)\,dx = 0 \qquad (i = 1, 2)$$

*for any function $\varphi \in L^2(0,T;W_0^{1,2}(\Omega)) \cap W^{1,2}(Q)$. In this case, the functions $u_i$ satisfy inequalities* (2).

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